Integration

Dr. K. M. Hock

Integrate is opposite of differentiate.

e.g. differentiate x^2 gives $2x$.

So integrate $2x$ gives x^2.

But differentiating x^2+1 also gives $2x$.

So integrating $2x$ can also give x^2+1.

If C is a constant (number that is the same for different x), then

differentiating x^2+C gives $2x$.

So integrating $2x$ gives x^2+C, where C is an unknown constant.

e.g. integrating $\cos x$ gives $\sin x + C$,

" $\frac{1}{x}$ " $\ln x + C$,

" e^x " $e^x + C$,

" $10x^9$ " $x^{10} + C$, and

" $-\sin x$ " $\cos x + C$.

$\boxed{\text{Must use radians for trigonometric functions.}}$

Integration of x^n, for any rational n, sin x, cos x , sec^2 x and e^x , together with constant multiples, sums and differences

Integration

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Symbol for integrating $2x$: $\int 2x\,dx = x^2 + C$.

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \qquad \int \sec^2 x\,dx = \tan x + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int e^x\,dx = e^x + C$$

$$\int \cos x\,dx = \sin x + C \qquad \int \frac{1}{x}\,dx = \ln x + C.$$

Multiple: $\int a f(x)\,dx = a\int f(x)\,dx$

e.g $\int 5\sin x\,dx = 5\int \sin x\,dx = -5\cos x + C$

Sum $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

e.g. $\int \left(\cos x + \frac{1}{x}\right)dx = \int \cos x\,dx + \int \frac{1}{x}\,dx$

$= \sin x + \ln x + C$

Difference $\int (f(x) - g(x))\,dx = \int f(x)\,dx - \int g(x)\,dx$

e.g. $\int (\sec^2 x + x^4)\,dx = \int \sec^2 x\,dx + \int x^4\,dx$

$= \tan x + \frac{1}{5}x^5 + C$

Integration

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Let $y = (ax+b)^n$. a, b are constants.

Let $u = ax+b$. So $y = u^n$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = nu^{n-1} \cdot a = na(ax+b)^{n-1}$$

So integrating this gives back $y = (ax+b)^n$:

$$\int na(ax+b)^{n-1}\, dx = (ax+b)^n + C$$

Divide by na: $\displaystyle\int (ax+b)^{n-1}\, dx = \frac{1}{na}(ax+b)^n + C$

Let $m = n-1$. Then $n = m+1$. So

$$\int (ax+b)^m\, dx = \frac{1}{(m+1)a}(ax+b)^{m+1} + C$$

Applying similar idea to the other functions:

$$\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int e^{ax+b}\, dx = \frac{1}{a}e^{ax+b} + C$$

Note: can also write $e^x = \exp(x)$ (just another symbol)

$e^{ax+b} = \exp(ax+b)$

Area under curve

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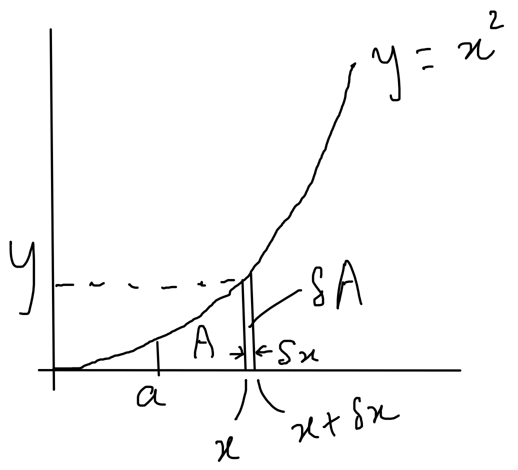
e.g.



Let A be area under curve between a and x.

If x increase a bit by δx, let δA be increase in A.

$\delta A \approx$ a rectangle

$\approx \underbrace{y}_{\text{height}} \underbrace{\delta x}_{\text{base}}$

Divide by δx:

$$\frac{\delta A}{\delta x} \approx y.$$

Gets more accurate as $\delta x \to 0$

$\because$ δA more like rectangle

But when $\delta x \to 0$, $\frac{\delta A}{\delta x} \to \frac{dA}{dx}$, derivative!

So $\frac{dA}{dx} = y$. Opposite is integral:

$$A = \int y\,dx = \int x^2\,dx = \frac{1}{3}x^3 + C.$$

e.g. if $a = 1$, find C.

ans. When $x = 1$, $A = 0$. So $0 = \frac{1}{3} \times 1^3 + C$

$C = -\frac{1}{3}$.

e.g. if $x = 3$, find A.

ans. $A = \frac{1}{3}x^3 + C = \frac{1}{3}(3)^3 - \frac{1}{3} = 8\frac{2}{3}$.

Definite Integral

Dr. K. M. Hock

e.g. $y = x^2$. Find area under curve between $x = 1$ and 3.

Ans. $A = \int y\,dx = \frac{1}{3}x^3 + C$.

when $x = 1$, $\quad 0 = \frac{1}{3}(1)^3 + C$

$x = 3$, $\quad \text{area} = \frac{1}{3}(3)^3 + C$

Subtracting, $\quad \text{area} = \frac{1}{3}(3)^3 - \frac{1}{3}(1)^3$

New symbol: $\quad = \left[\frac{1}{3}x^3\right]_1^3$

Another symbol: $\int_1^3 x^2\,dx = \left[\frac{1}{3}x^3\right]_1^3$.

Rewrite answer:

$$\text{area} = \int_1^3 x^2\,dx = \left[\frac{1}{3}x^3\right]_1^3 = \frac{1}{3}(3)^3 - \frac{1}{3}(1)^3 = 8\tfrac{2}{3}$$

Let $y = f(x)$. Let $\int f(x)\,dx = F(x) + C$

Definite Integral

$$\boxed{\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)}$$

e.g. find $\int_{\pi/6}^{\pi/3} \cos x\,dx$.

Ans. $\int_{\pi/6}^{\pi/3} \cos x\,dx = \left[\sin x\right]_{\pi/6}^{\pi/3}$

$$= \left(\sin\frac{\pi}{3} - \sin\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}$$

Must use radians for trigonometric functions.

Finding the area of a region bounded by a curve and line(s) (excluding area of region between two curves)

Bounded Region

Dr K M Hock

e.g. Find the area of the region bounded by $y = -x^2 + 6x$ and $y = x + 4$.

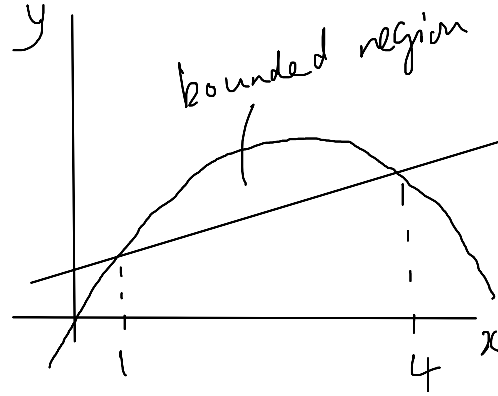
ans. Sketch the graphs. Find the points of intersections.

Substituting, $x + 4 = -x^2 + 6x$

$$x^2 - 5x + 4 = 0$$

$$(x-1)(x-4) = 0$$

$$x = 1, 4$$



So area is between these.

bounded Area $= $ area under Curve $-$ area under line

$$= \int_1^4 (-x^2 + 6x)\,dx - \int_1^4 (x+4)\,dx$$

$$= \int_1^4 (-x^2 + 5x - 4)\,dx$$

$$= \left[-\frac{1}{3}x^3 + \frac{5}{2}x^2 - 4x\right]_1^4$$

$$= \left[-\frac{1}{3}(4^3) + \frac{5}{2}(4^2) - 4(4)\right] - \left[-\frac{1}{3} + \frac{5}{2} - 4\right]$$

$$= \underline{\qquad}.$$

Area below x-axis

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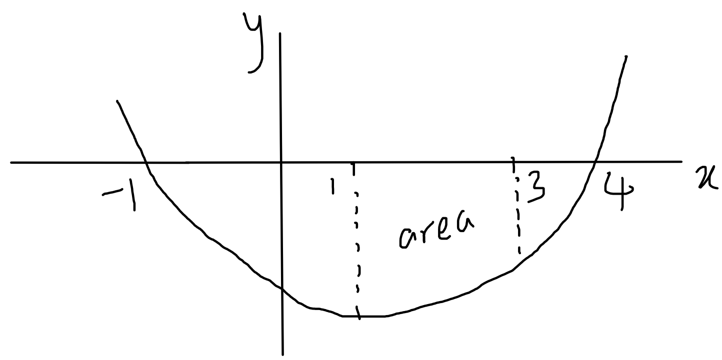
e.g. Find the area between x-axis and the curve $y = x^2 - 3x - 4$ between $x = 1$ and $x = 3$.

ans. Check if curve intersects x-axis:

$$0 = x^2 - 3x - 4$$
$$0 = (x+1)(x-4)$$

x intercepts at -1 and 4.



So the area is below x-axis.

Can still use integration, but answer is negative:

$$\int_1^3 (x^2 - 3x - 4)\,dx = \left[\frac{1}{3}x^3 - \frac{3}{2}x^2 - 4x \right]_1^3$$
$$= \left[\frac{1}{3}(3)^3 - \frac{3}{2}(3^2) - 4(3)\right] - \left[\frac{1}{3} - \frac{3}{2} - 4\right]$$
$$= -11\tfrac{1}{3}$$

Take positive value: area $= 11\frac{1}{3}$.

(Actually, just reflecting curve in x-axis.)

Application of differentiation and integration to problems involving displacement, velocity and acceleration of a particle moving in a straight line

Acceleration of Particle

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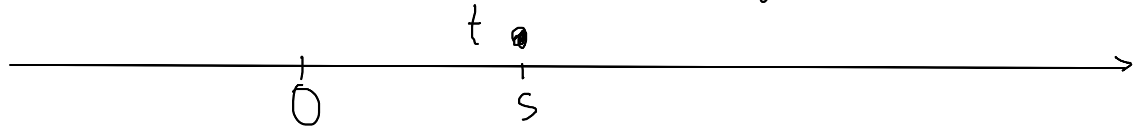
A particle moves on a straight line.

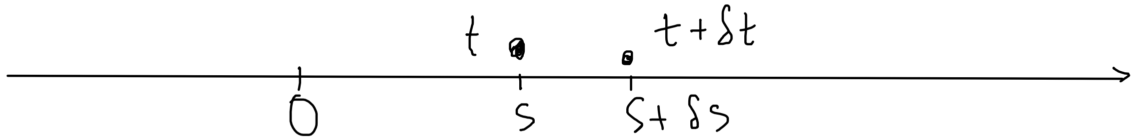


At a time t, it is at a distance s to the right of a point O.

We give s a +ve sign if it is right of O,
a −ve " " " " left " O.

We call s the displacement from O.

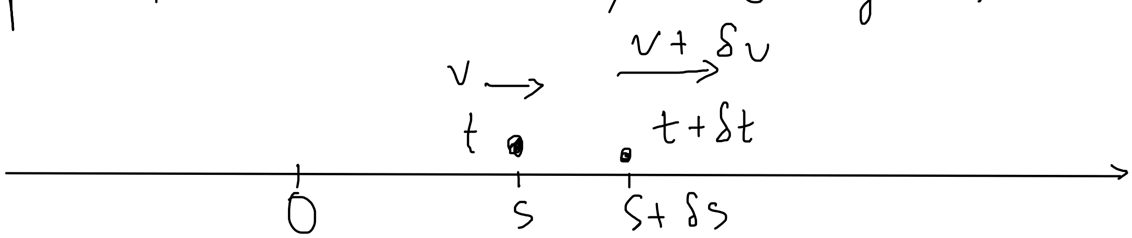
After a time δt, the particle moves to $s + \delta s$



average velocity over δt is $\dfrac{\text{displacement}}{\text{time taken}} = \dfrac{\delta s}{\delta t}$

If $\delta t \to 0$, we get the velocity v at t: $\dfrac{\delta s}{\delta t} \to \dfrac{ds}{dt} = v$

Suppose that at $t + \delta t$, v changes to $v + \delta v$.



Then the average acceleration $= \dfrac{\text{change in velocity}}{\text{time taken}} = \dfrac{\delta v}{\delta t}$

When $\delta t \to 0$, we get the acceleration a at t: $\dfrac{\delta v}{\delta t} \to \dfrac{dv}{dt} = a$.

Problem 1

Dr. K. M. Hock

2013P2Q9 A particle, moving in a straight line, passes through a fixed point O with a speed of 28 m/s. The acceleration a m/s^2 of the particle, t s after passing through O, is given by $a = -16e^{-0.5t}$.

(i) Find the value of t when the particle is at instantaneous rest.

(ii) Find the distance of the particle from O when it is at instantaneous rest.

Solution.

(i) <u>Instantaneous rest</u> means velocity $= 0$ at a particular time, but can be different before or after.

Let $s =$ displacement. Then velocity $v = \frac{ds}{dt}$,
acceleration $a = \frac{dv}{dt}$.

Given that $\frac{dv}{dt} = a = -16e^{-0.5t}$

So $v = \int a\,dt = \int -16e^{-0.5t}\,dt = 32e^{-0.5t} + C$

At $t = 0$, $v = 28$. So $28 = 32 + C$, $C = -4$.

$\therefore v = 32e^{-0.5t} - 4$

When particle is at rest, $0 = 32e^{-0.5t} - 4$
$e^{0.5t} = 8$
$0.5t = \ln 8$, $t = 4.159$

(ii) $\frac{ds}{dt} = v \Rightarrow s = \int v\,dt = \int (32e^{-0.5t} - 4)\,dt = 64e^{-0.5t} - 4t + C$

When $t = 0$, $s = 0$. So $0 = 64 + C$ $\therefore$ when $t = 4.159$, $s =$ ___.

Problem 2

Dr. K. M. Hock

2013P1Q12 (i) Express $\frac{2x}{2x+3}$ in the form $a + \frac{b}{2x+3}$, where a and b are integers.

(ii) Differentiate $x\ln(2x+3)$ with respect to x.

(iii) Using the results of parts (i) and (ii), determine $\int \ln(2x+3)\,dx$.

Solution.

(i) $$\frac{2x}{2x+3} = \frac{2x+3-3}{2x+3} = 1 - \frac{3}{2x+3}$$

(ii) Let $y = x\ln(2x+3)$. Let $u = x$, $v = \ln(2x+3)$.

So $y = uv$. $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= x \cdot \frac{2}{2x+3} + \ln(2x+3).$$

(iii) From (ii), $$\int \frac{2x}{2x+3}\,dx + \int \ln(2x+3)\,dx = y$$

$$\int \ln(2x+3)\,dx = y - \int \frac{2x}{2x+3}\,dx$$

$$= y - \int \left(1 - \frac{3}{2x+3}\right) dx \qquad \text{from (i)}$$

$$= y - \left(x - \tfrac{3}{2}\ln(2x+3)\right) + C$$

$$= x\ln(2x+3) - x + \tfrac{3}{2}\ln(2x+3) + C$$

Problem 3   Dr. K. M. Hock

2013 P2 Q5 (i) Express $3\cos^2 x - \sin^2 x$ in the form $a + b\cos 2x$, where a and b are constants to be found.

(ii) Using your values of a and b, find $\int (3\cos^2 x - \sin^2 x)\,dx$ and hence evaluate

$$\int_{-\pi/12}^{\pi/12} (3\cos^2 x - \sin^2 x)\,dx.$$

Solution.

(i) Identities:

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\cos 2x = 2\cos^2 x - 1$$

$$\begin{aligned} \therefore\ 3\cos^2 x - \sin^2 x &= 2\cos^2 x + \cos^2 x - \sin^2 x \\ &= \cos 2x + 1 + \cos 2x \\ &= 2\cos 2x + 1. \end{aligned}$$

(ii)

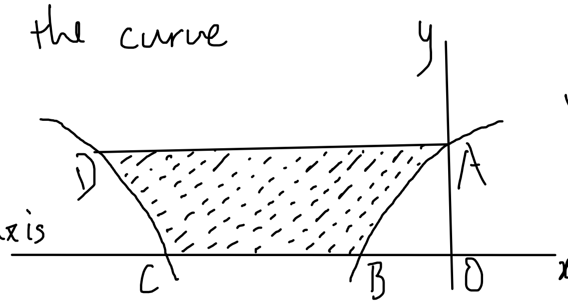
$$\begin{aligned} \int (3\cos^2 x - \sin^2 x)\,dx &= \int (2\cos 2x + 1)\,dx \\ &= \sin 2x + x + C. \end{aligned}$$

$$\begin{aligned} \int_{-\pi/12}^{\pi/12} (3\cos^2 x - \sin^2 x)\,dx &= \Big[\sin 2x + x\Big]_{-\pi/12}^{\pi/12} \\ &= \left(\sin\frac{\pi}{6} + \frac{\pi}{12}\right) - \left(\sin\left(-\frac{\pi}{6}\right) - \frac{\pi}{12}\right) \\ &= \left(\frac{1}{2} + \frac{\pi}{12}\right) - \left(-\frac{1}{2} - \frac{\pi}{12}\right) \\ &= 1 + \frac{\pi}{6}. \end{aligned}$$

Problem 4

Dr. K. M. Hock

2013 P2Q11 The diagram shows parts of the curve $y = 3 - \frac{12}{(x+3)^2}$ intersecting the y axis at the point A and intersecting the x-axis at the points B and C.



The point D lies on the curve and AD is parallel to the x-axis. Find

(i) the coordinates of A, B, C and D,

(ii) the area bounded by the curve AB and the coordinate axes,

(iii) the area of the shaded region.

Solution

(i) A: $x = 0$, $y = 3 - \frac{12}{3^2} = \frac{5}{3}$

D: $y = \frac{5}{3}$, $(x+3)^2 = (0+3)^2 \Rightarrow x = -6$.

B, C: $y = 0$, $3 = \frac{12}{(x+3)^2} \Rightarrow (x+3)^2 = 4$,

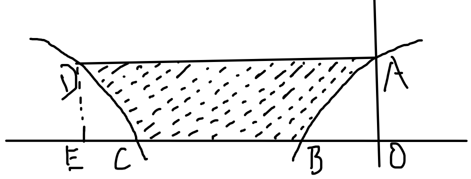
$x = -1, -5$.

$\therefore A(0, \frac{5}{3})$, $B(-1, 0)$, $C(-5, 0)$, $D(-6, \frac{5}{3})$.

(ii) $$\int_B^A y\,dx = \int_{-1}^{0} \left(3 - \frac{12}{(x+3)^2}\right) dx = \left[3x + \frac{12}{x+3}\right]_{-1}^{0}$$

$$= \left[0 + \frac{12}{3}\right] - \left[-3 + \frac{12}{-1+3}\right] = 4 - (-3+6) = 1$$

(iii) $y = 3 - \frac{12}{(x+3)^2}$ is symmetrical about $x = -3$. (e.g. $x = -3+2$ or $-3-2$ give same y.)



So DEC same area as AOB.

Shaded area = rectangle $OADE$ − DEC − AOB

$= \frac{5}{3} \times 6 - 1 - 1 = 8$